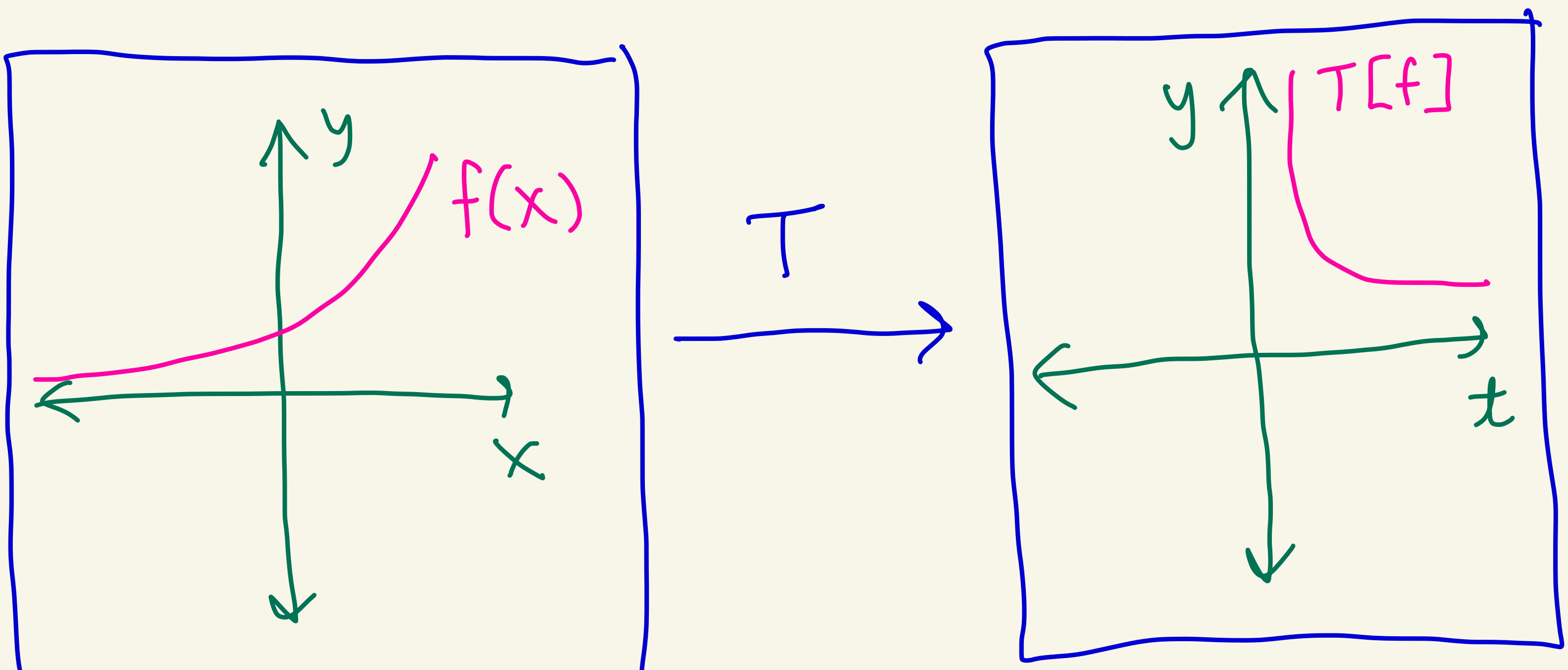
Math 2150

Topic 14 - Laplace Transforms

In mathematics an integral transform is a function T that takes a function f and transforms it into $T[f]$ using the formula:

$$T[f](t) = \int_{x_1}^{x_2} \underbrace{K(x,t)}_{\text{called the kernel of the transform}} f(x)\,dx$$

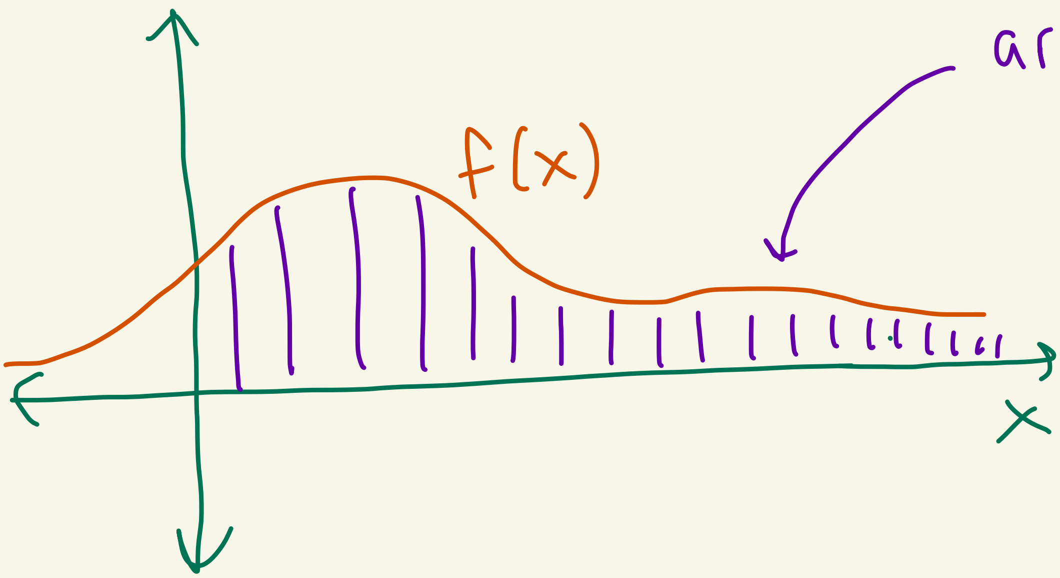


Examples

Name	Kernel
Fourier transform	$e^{-2\pi i t x}$
Laplace transform	e^{-tx}
Mellin transform	x^{t-1}
$\vdots$	$\vdots$

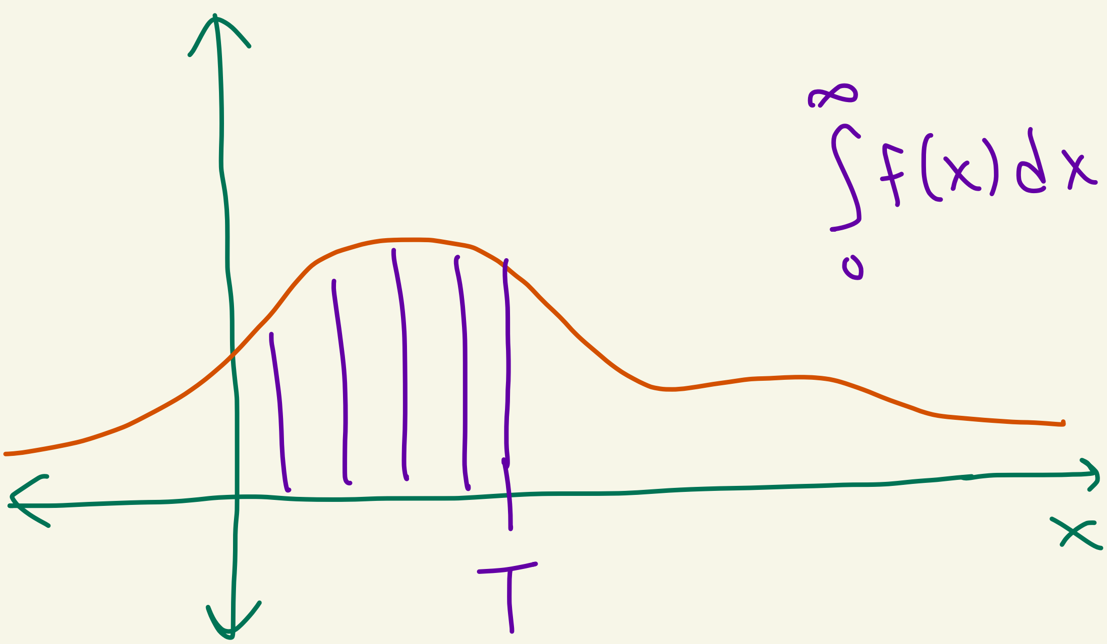
For the Laplace transform we need improper integrals.

Recall:



$$\text{area} = \int_0^\infty f(x)\,dx$$

which is defined as



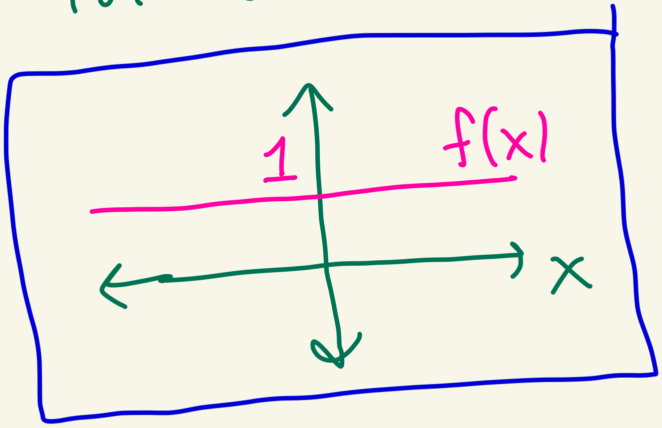
$$\int_0^\infty f(x)\,dx = \lim_{T\to\infty} \int_0^T f(x)\,dx$$

Def: Given a function $f(x)$ defined for $0 \le x < \infty$, let the Laplace transform of f be:

$$\mathcal{L}[f] = \int_0^\infty \underbrace{e^{-tx}}_{\boxed{\text{Kernel}}} f(x)\,dx$$

Note: $\mathcal{L}[f]$ is a function of t. So we sometimes write $\mathcal{L}[f](t)$.

Ex: Let $f(x) = 1$ for all x.



Then,

$$\mathcal{L}[f] = \int_0^\infty e^{-tx} f(x)\,dx$$

$$= \int_0^\infty e^{-tx} \cdot 1\,dx = \boxed{\int_0^\infty e^{-tx}\,dx}$$

Let's try various t values.

$$\mathcal{L}[f](2) = \int_0^\infty e^{-2x}\,dx$$

$t=2$

$$= \lim_{T\to\infty} \int_0^T e^{-2x}\,dx$$

$$= \lim_{T\to\infty} \left[-\frac{1}{2} e^{-2x} \Big|_0^T \right]$$

$$= \lim_{T\to\infty}\left[-\frac{1}{2}e^{-2T} + \frac{1}{2}e^{-2(0)}\right]$$

$$= \lim_{T\to\infty}\left[-\frac{1}{2}\cdot\frac{1}{e^{2T}} + \frac{1}{2}\right]$$

$$= \left[0 + \frac{1}{2}\right] = \frac{1}{2}$$

Let's try $t=-1$.

We get

$$\mathcal{L}[f](-1) = \int_0^\infty e^{-(-1)x}\,dx$$

$$= \lim_{T\to\infty}\int_0^T e^x\,dx$$

$$= \lim_{T\to\infty}\left[e^T - e^0\right] = \infty$$

So, $\mathcal{L}[f](-1)$ is undefined.

It turns out that $\mathcal{L}[f](t)$ when $f(x) = 1$ is undefined when $t \leq 0$.

But if $t > 0$ then we get

$$\mathcal{L}[f](t) = \int_0^\infty e^{-tx}\,dx$$

$$= \lim_{T \to \infty} \int_0^T e^{-tx}\,dx$$

$$= \lim_{T \to \infty} \left[-\frac{1}{t} e^{-tx} \Big|_{x=0}^{T} \right]$$

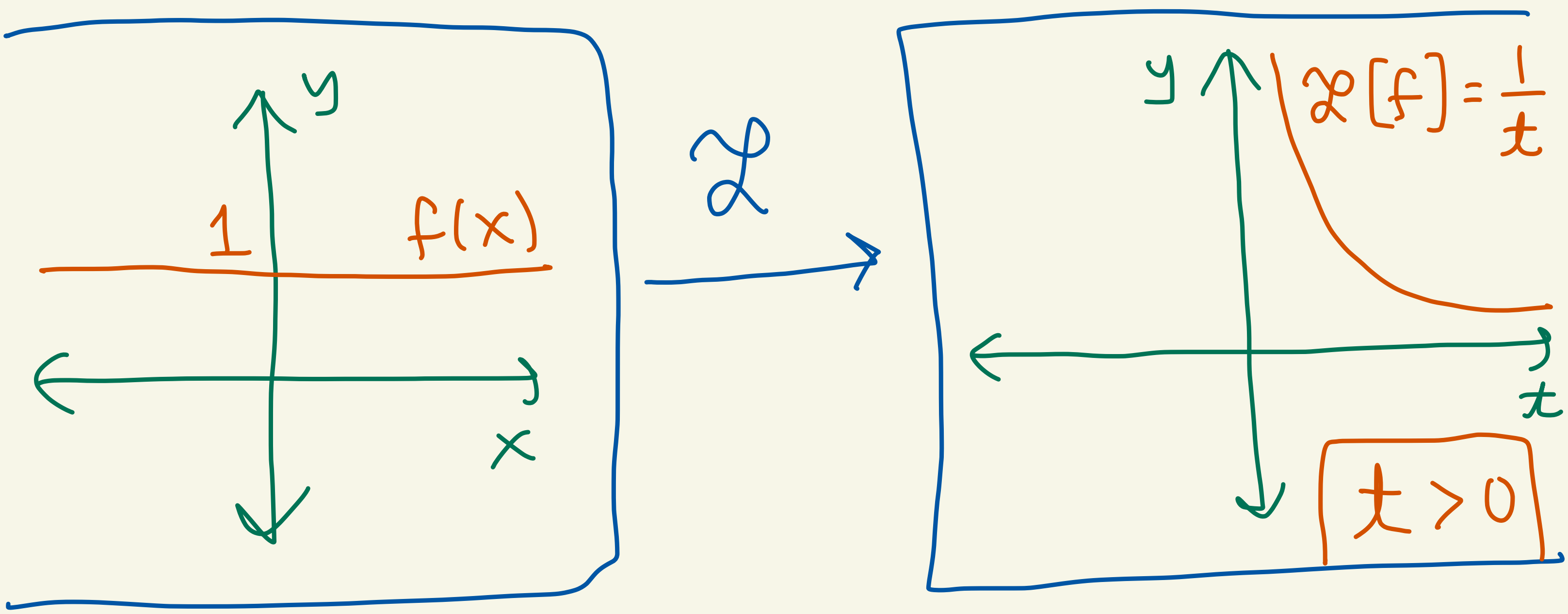
$$= \lim_{T \to \infty} \left[-\frac{1}{t} e^{-tT} + \frac{1}{t} e^{0} \right]$$

$$= \lim_{T \to \infty} \left[-\frac{1}{t} \cdot \frac{1}{e^{tT}} + \frac{1}{t} \right]$$

as $T \to \infty$ because $t > 0$

0

$$= 0 + \frac{1}{t} = \frac{1}{t}$$



Ex: Let $f(x) = e^{ax}$ where a is any constant.

If $t > a$, then

$$\mathcal{L}[f] = \int_0^\infty e^{-tx} f(x)\,dx$$

$$= \int_0^\infty e^{-tx} e^{ax}\,dx$$

$$= \lim_{T\to\infty} \int_0^T e^{(a-t)x}\,dx$$

$$= \lim_{T\to\infty} \left[\frac{1}{(a-t)} e^{(a-t)x} \Big|_{x=0}^{T} \right]$$

$$= \lim_{T\to\infty} \left[\frac{1}{(a-t)} e^{(a-t)T} - \frac{1}{(a-t)} e^{0} \right]$$

$t > a$

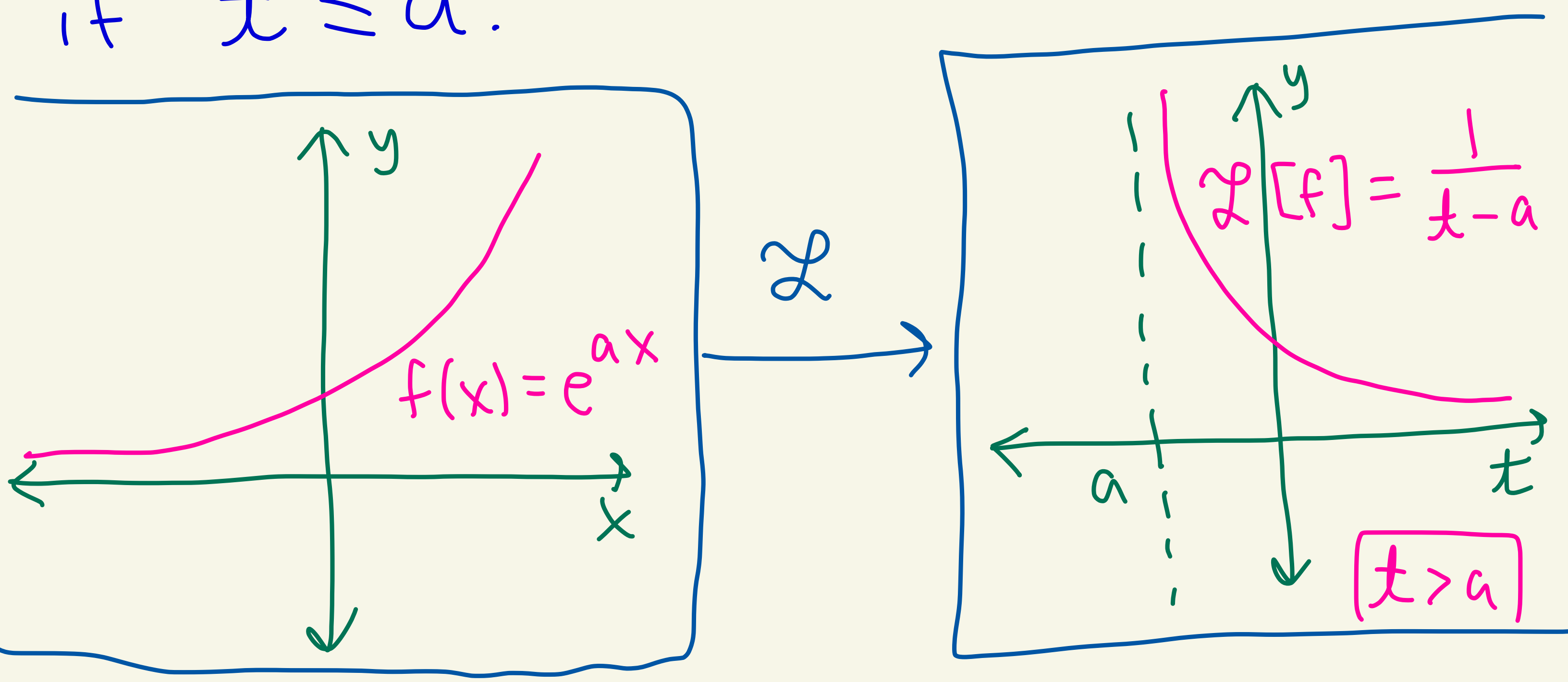
$0 > a - t$

0 as $T \to \infty$

$$= \left[0 - \frac{1}{a-t} \cdot 1 \right]$$

$$= \frac{1}{t-a}$$

The above integral would diverge if $t \leq a$.



Some Laplace Transforms

① $\mathcal{L}[1] = \frac{1}{t}$ where $t > 0$

② $\mathcal{L}[x^n] = \frac{n!}{t^{n+1}}$ where $t > 0$
$n = 1, 2, 3, \ldots$

③ $\mathcal{L}[e^{ax}] = \frac{1}{t-a}$ where $t > a$

④ $\mathcal{L}[\sin(kx)] = \frac{k}{t^2+k^2}$ where $t > 0$

⑤ $\mathcal{L}[\cos(kx)] = \frac{t}{t^2+k^2}$ where $t > 0$

Theorem:

Suppose $\mathcal{L}[f]$ and $\mathcal{L}[g]$ both exist for $t > t_0$.

Then,

$$\mathcal{L}[c_1 f + c_2 g] = c_1 \mathcal{L}[f] + c_2 \mathcal{L}[g]$$

when $t > t_0$.

Here c_1, c_2 are constants.

Furthmore, if f, f', f'' are continuous on $[0, \infty)$ and their Laplace transforms exist then

$$\mathcal{L}[f'] = t \cdot \mathcal{L}[f] - f(0)$$

$$\mathcal{L}[f''] = t^2 \mathcal{L}[f] - t f(0) - f'(0)$$

Ex: Solve

$$\boxed{\begin{array}{l} y' + 2y = e^{-x} \\ y(0) = 2 \end{array}}$$

using Laplace transforms.

Suppose the laplace transform exists for the solution of the above.

Let $Y(t) = \mathcal{L}[y]$.

Then,

$$\mathcal{L}[y' + 2y] = \mathcal{L}[e^{-x}]$$

$$\mathcal{L}[y'] + 2\mathcal{L}[y] = \frac{1}{t+1}$$

$$\left(t\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] = \frac{1}{t+1}$$

$$tY(t) - 2 + 2Y(t) = \frac{1}{t+1}$$

$$(t+2)Y(t) = \frac{1}{t+1} + 2$$

$$Y(t) = \frac{1}{(t+1)(t+2)} + \frac{2}{t+2}$$

$$Y(t) = \frac{1}{t+1} - \frac{1}{t+2} + \frac{2}{t+2}$$

$$\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$

$$1 = A(t+2) + B(t+1)$$

$\underline{t=-2}$: $1 = A(0) + B(-1)$

$B = -1$

$\underline{t=-1}$: $1 = A(1) + B(0)$

$1 = A$

$$y(t) = \frac{1}{t+1} + \frac{1}{t+2}$$

We need some function y whose Laplace transform satisfies

$$\mathcal{L}[y] = \frac{1}{t+1} + \frac{1}{t+2}$$

Using the table we have

$$y(x) = e^{-x} + e^{-2x}$$

$$L\left[e^{-x} + e^{-2x}\right] = L\left[e^{-x}\right] + L\left[e^{-2x}\right] = \frac{1}{t+1} + \frac{1}{t+2}$$

So the solution is

$$\boxed{y(x) = e^{-x} + e^{-2x}}$$

Ex: Solve

$$y'' + 4y = 5e^{-x}$$
$$y(0) = 2, \quad y'(0) = 3$$

using Laplace transforms.

Suppose the Laplace transform exists for the solution of the above.

Let $Y(t) = \mathcal{L}[y]$.

Then,

$$\mathcal{L}[y'' + 4y] = \mathcal{L}[5e^{-x}]$$
$$\mathcal{L}[y''] + 4\mathcal{L}[y] = 5\mathcal{L}[e^{-x}]$$
$$\left(t^2\mathcal{L}[y] - t\,y(0) - y'(0)\right) + 4\mathcal{L}[y] = 5 \cdot \frac{1}{t+1}$$
$$t^2 Y(t) - 2t - 3 + 4Y(t) = \frac{5}{t+1}$$
$$Y(t)\left[t^2 + 4\right] = \frac{5}{t+1} + 2t + 3$$
$$Y(t) = \frac{5}{(t^2+4)(t+1)} + \frac{2t+3}{t^2+4}$$

$$\frac{5}{(t^2+4)(t+1)} = \frac{At+B}{t^2+4} + \frac{C}{t+1}$$
$$5 = (At+B)(t+1) + C(t^2+4)$$
$\underline{t=-1:}$ $5 = 0 + C(5) \to C = 1$

$\underline{t=0:}$ $5 = B(1) + (1)(4)$
$1 = B$

$\underline{t=1:}$ $5 = (A+1)(2) + (1)(5)$
$-1 = A$

$$Y(s) = \left(\frac{-t}{t^2+4} + \frac{1}{t^2+4} + \frac{1}{t+1}\right) + \left(\frac{2t}{t^2+4} + \frac{3}{t^2+4}\right)$$

$$Y(t) = 4\frac{1}{t^2+4} + \frac{1}{t+1} + \frac{t}{t^2+4}$$

What $y(x)$ has $\mathcal{L}[y]$ equal to the above?

$$\boxed{\text{The answer is: } \quad y(x) = 2\sin(2x) + e^{-x} + \cos(2x)}$$

Check:

$$\mathcal{L}\left[2\sin(2x) + e^{-x} + \cos(2x)\right]$$

$$= 2\left(\frac{2}{t^2+2^2}\right) + \frac{1}{t-(-1)} + \frac{t}{t^2+2^2}$$

$$= 4\,\frac{1}{t^2+4} + \frac{1}{t+1} + \frac{t}{t^2+4}$$